STABILITY OF SINGLE-FREQUENCY PERIODIC SOLUTIONS OF QUASI-LINEAR SELF-CONTAINED SYSTEMS WITH TWO DEGREES OF FREEDOM

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PMM Vol.29, № 5, 1965, pp.939-945

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(Received April 13, 1965)

1. Let us consider a quasi-linear self-contained system with two degrees of freedom of the form

$$x_i^{\cdot\cdot} + \omega_i^2 x_i = \mu F_i(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}, \mu) \qquad (i = 1, 2) \tag{1.1}$$

The function F_i is analytic in its arguments and μ is a small parameter. It is assumed that the frequencies ω_1 and ω_2 are incommensurable.

As is known [1], the system (1.1) can be obtained by means of a linear transformation from a quasi-linear system of general form, the generating system ($\mu = 0$) of which is a conservative system with potential energy represented in a difinite positive quadratic form.

The initial conditions are assumed as follows [1] :

$$x_1(0) = A_0 + \beta, \qquad x_1^{\cdot}(0) = 0, \qquad x_2(0) = \varphi, \qquad x_2^{\cdot}(0) = \psi \tag{1.2}$$

The quantities β, φ and ψ are functions of μ which are expandable in integral, and β in some cases fractional powers of the parameter, and vanishing for $\mu = 0$. Furthermore, it is assumed that φ and ψ depend also on $A_0 + \beta$. Thus, we have

$$\varphi(A_0 + \beta, \mu) = \sum_{n=1}^{\infty}\left[p_n(A_0) + \frac{dp_n}{dA_0}\beta + \cdots\right]\mu^n$$

$$\psi(A_0 + B, \mu) = \sum_{n=1}^{\infty}\left[q_n(A_0) + \frac{dq_n}{dA_0}\beta + \cdots\right]\mu^n$$

The periodic solution of the system (1.1) with period $T_1^* = 2\pi / \omega_1 + \alpha^*$ can be expressed in the form

$$x_1(t) = (A_0 + \beta)\cos\omega_1 t + \sum_{n=1}^{\infty}\left[C_{1n}(t) + \frac{dC_{1n}(t)}{dA_0}\beta + \frac{1}{2}\frac{d^2C_{1n}(t)}{dA_0^2}\beta^2 + \cdots\right]\mu^n$$

$$x_2(t) = \varphi\cos\omega_2 t + \frac{\psi}{\omega_2}\sin\omega_2 t + \sum_{n=1}^{\infty}\left[C_{2n}(t) + \frac{dC_{2n}(t)}{dA_0}\beta + \cdots\right]\mu^n \tag{1.3}$$

The functions $C_{in}(t)$ depend on all initial conditions and, consequently, are composite functions of $A_0+\beta$. The derivatives of these functions with respect to A_0 are computed by taling th s fact into consideration. The functions $C_{in}(t)$ for $\beta=0$ are determined according to Formulas

$$C_{in}(t)=\frac{1}{\omega_i}\int_0^t H_{in}(t_1)\sin\omega_i(t-t_1)\,dt_1, \qquad H_{in}(t)=\frac{1}{(n-1)!}\left(\frac{d^{n-1}F_i}{d\mu^{n-1}}\right)_{\beta=\mu=0} \tag{1.4}$$

Let us introduce new functions

$$D_n(t)=C_{2n}(t)+p_n\cos\omega_2 t+\frac{q_n}{\omega_2}\sin\omega_2 t$$

Then the values of the quantities $H_{in}(t)$ in the expanded form will be

$$H_{i1}(t)=F_i(x_{10},\,x_{20},\,x_{10}^{\cdot},\,x_{20}^{\cdot},\,0)$$

$$H_{i2}(t)=\left(\frac{\partial F_i}{\partial x_1}\right)_0 C_{11}(t)+\left(\frac{\partial F_i}{\partial x_2}\right)_0 D_1(t)+\left(\frac{\partial F_i}{\partial x_1^{\cdot}}\right)_0 C_{11}^{\cdot}(t)+\left(\frac{\partial F_i}{\partial x_2^{\cdot}}\right)_0 D_1^{\cdot}(t)+\left(\frac{\partial F_i}{\partial \mu}\right)_0 \quad \text{etc.} \tag{1.5}$$

The subscript 0 in the F_i derivatives indicates that for $\mu=0$, instead of x_1, x_2, $x_1^{\cdot}$, $x_2^{\cdot}$ and μ, the values of these quantities, i.e. x_{10}, x_{20}, $x_{10}^{\cdot}$, $x_{20}^{\cdot}$ and 0 should be substituted into the derivatives.

In order to represent the solution as a series in μ with coefficients whose period $T_1=2\pi/\omega_1$ is independent of μ, the transformation of time t is utilized. To retain the solution in the form (1.3), the coefficients of this transformation must be functions of $A_0+\beta$. Therefore, the transformation will be in the form

$$t=\tau\left[1+\frac{1}{T_1}\sum_{n=1}^{\infty}N_n(A_0+\beta)\mu^n\right]=h\tau \tag{1.6}$$

For a system with a single degree of freedom the values of the coefficients N_n for $\beta=0$ are given in [2]. In the present case we have

$$N_1=\frac{1}{A_0\omega_1^2}C_{11}^{\cdot}(T_1), \qquad N_2=\frac{1}{A_0\omega_1^2}[C_{12}^{\cdot}(T_1)+N_1C_{11}^{\cdot\cdot}(T_1)] \quad \text{etc.} \tag{1.7}$$

Using the substitution (1.6) we get

$$x_i(t)=x_i(h\tau)=z_i(\tau), \qquad x^{\cdot}(t)=z_i'(\tau)\frac{1}{h}$$

The system (1.1) becomes

$$z_i''+h^2\omega_i^2 z_i=\mu h^2 F_i\left(z_1,\,z_2,\,z_1'\frac{1}{h},\,z_2'\frac{1}{h},\,\mu\right)$$

This system can also be expressed as

$$z_i''+\omega_i^2 z_i=\mu\Phi_i(z_1,\,z_2,\,z_1',\,z_2',\,\mu) \qquad (i=1,2) \tag{1.8}$$

Here

$$\Phi_i(z_1,\,z_2,\,z_1',\,z_2',\,\mu)=h^2F_i\left(z_1,\,z_2,\,z_1'\frac{1}{h},\,z_2'\frac{1}{h},\,\mu\right)-\frac{h^2-1}{\mu}\omega_i^2 z_i \tag{1.9}$$

The solution of the system (1.8) can be expressed in the same form as that of the system (1.1)

$$z_1(\tau)=(A_0+\beta)\cos\omega_1\tau+\sum_{n=1}^{\infty}\left[C_{1n}^*(\tau)+\frac{dC_{1n}^*(\tau)}{dA_0}\beta+\frac{1}{2}\frac{d^2C_{1n}^*(\tau)}{dA_0^2}\beta^2+\cdots\right]\mu^n$$

$$z_2(\tau)=\varphi\cos\omega_2\tau+\frac{\psi}{\omega_2}\sin\omega_2\tau+\sum_{n=1}^{\infty}\left[C_{2n}^*(\tau)+\frac{dC_{2n}^*(\tau)}{dA_0}\beta+\cdots\right]\mu^n \tag{1.10}$$

Introducing the notation

$$D_n^*(\tau)=C_{2n}^*(\tau)+p_n\cos\omega_2\tau+\frac{q_n}{\omega_2}\sin\omega_2\tau \qquad H_{in}^*(\tau)=\frac{1}{(n-1)!}\left(\frac{d^{n-1}\Phi_i}{d\mu^{n-1}}\right)_{\beta=\mu=0}$$

then for the quantities $H_{in}^*(\tau)$ the formulas analogous to (1.5) will take place

$$H_{i1}^*(\tau) = \Phi_i(z_{10}, z_{20}, z_{10}', z_{20}', 0)$$

$$H_{i2}^*(\tau) = \left(\frac{\partial \Phi_i}{\partial z_1}\right)_0 C_{11}^*(\tau) + \left(\frac{\partial \Phi_i}{\partial z_2}\right)_0 D_1^*(\tau) + \left(\frac{\partial \Phi_i}{\partial z_1'}\right)_0 C_{11}^{*\prime}(\tau) + \\ + \left(\frac{\partial \Phi_i}{\partial z_2'}\right)_0 D_1^{*\prime}(\tau) + \left(\frac{\partial \Phi_i}{\partial \mu}\right)_0 \quad \text{etc.}$$

The functions $C_{1n}^*(\tau)$ and $C_{1n}(\tau)$ are related as follows:

$$C_{11}^*(\tau) = C_{11}(\tau) - \frac{1}{T_1} N_1 A_0 \omega_1 \tau \sin \omega_1 \tau$$

$$C_{12}^*(\tau) = C_{12}(\tau) + \frac{1}{T_1} N_1 \tau C_{11}'(\tau) - \frac{1}{T_1} N_2 A_0 \omega_1 \tau \sin \omega_1 \tau - \frac{1}{2}\frac{1}{T_1^2} N_1^2 A_0 \omega_1^2 \tau^2 \cos \omega_1 \tau \tag{1.11}$$

etc.

Analogously, for the functions $C_{2n}^*(\tau)$ as well as for the quantities $H_{in}^*(\tau)$ the corresponding formulas can be written. It is not difficult to show that

$$C_{1n}^*(T_1) = M_n \tag{1.12}$$

The values of the quantities M_n for a single degree of freedom are given in [2]. In the present case

$$M_1 = C_{11}(T_1), \qquad M_2 = C_{12}(T_1) + \frac{1}{2A_0\omega_1^2} C_{11}'^2(T_1) \quad \text{etc.} \tag{1.13}$$

Proceeding from the initial condition $z_1'(0) = 0$, it is easy to see that

$$C_{1n}^{*\prime}(T_1) = 0 \tag{1.14}$$

Since this equality is fulfilled identically, the derivatives of any order of $C_{1n}^{*\prime}(T_1)$ with respect to A_0 are also equal to zero.

2. Let us investigate the stability of the periodic solutions of system (1.8) with period T_1 with the assumption that ω_1 and ω_2 are incommensurable. We will consider the cases when the equation defining the amplitude A_0

$$C_{11}^*(T_1) = C_{11}(T_1) = 0 \tag{2.1}$$

has not only simple, but also double and triple roots.

We construct the equations in variations for the system (1.8). We have

$$y_i'' + \omega_i^2 y_i = \mu\left[\frac{\partial \Phi_i}{\partial z_1} y_1 + \frac{\partial \Phi_i}{\partial z_2} y_2 + \frac{\partial \Phi_i}{\partial z_1'} y_1' + \frac{\partial \Phi_i}{\partial z_2'} y_2'\right] \qquad (i = 1, 2) \tag{2.2}$$

At the same time it is assumed that in the functions Φ_i was substituted the periodic solution $z_1(\tau)$ and $z_2(\tau)$ with the period $T_1 = 2\pi/\omega_1$. The solution of the equations in variations will be sought in the form [3]

$$y_1(\tau) = e^{\alpha\tau} u(\tau), \qquad y_2(\tau) = e^{\alpha\tau} v(\tau) \tag{2.3}$$

where $u(\tau)$ and $v(\tau)$ are periodic functions of τ with period T_1, while α is the characteristic exponent. Substituting (2.3) for $y_1(\tau)$ and $y_2(\tau)$ in Equations (2.2), we will obtain the equations for determination of the functions $u(\tau)$ and $v(\tau)$ as well as the characteristic exponent α

$$u'' + 2\alpha u' + (\alpha^2 + \omega_1^2) u = \mu\left[\left(\frac{\partial \Phi_1}{\partial z_1} + \alpha \frac{\partial \Phi_1}{\partial z_1'}\right) u + \frac{\partial \Phi_1}{\partial z_1'} u' + \left(\frac{\partial \Phi_1}{\partial z_2} + \alpha \frac{\partial \Phi_1}{\partial z_2'}\right) v + \frac{\partial \Phi_1}{\partial z_2'} v'\right]$$

$$v'' + 2\alpha v' + (\alpha^2 + \omega_2^2) v = \mu\left[\left(\frac{\partial \Phi_2}{\partial z_1} + \alpha \frac{\partial \Phi_2}{\partial z_1'}\right) u' + \frac{\partial \Phi_2}{\partial z_1'} u' + \left(\frac{\partial \Phi_2}{\partial z_2} + \alpha \frac{\partial \Phi_2}{\partial z_2'}\right) v + \frac{\partial \Phi_2}{\partial z_2'} v'\right] \tag{2.4}$$

The functions $u(\tau)$ and $v(\tau)$, as well as the characteristic exponent α can be expanded in series in integral or fractional powers of μ.

Since the frequencies of the generating system are equal to ω_1 and ω_2, the roots of the corresponding fundamental equation [3] are equal to $\pm i\omega_1$

and $\pm i\omega_2$. For a periodic solution of the system (1.8), with period T_1, the first two roots are critical and the second not critical.

3. Let us comp te the characteristic exponent $\alpha^{(1)}$ for the critical roots. Note that in expanding this exponent in μ the constant term can be omitted. This follows from the fact that the quantity

$$e^{\alpha_0\tau} = e^{\pm i\omega_1\tau} = \cos\omega_1\tau \pm i\sin\omega_1\tau$$

is a periodic function of τ with period T_1 and, consequently, can be included in the function $u(\tau)$ and $v(\tau)$.

Let us first consider the case when the periodic solution of the generating system is expanded in power series in $\mu^{1/2}$. We have

$$\alpha^{(1)} = \alpha_{1/2}\mu^{1/2} + \alpha_1\mu + \alpha_{3/2}\mu^{3/2} + \cdots$$

$$u^{(1)}(\tau) = u_0(\tau) + \mu^{1/2}u_{1/2}(\tau) + \mu u_1(\tau) + \cdots \tag{3.1}$$

$$v^{(1)}(\tau) = v_0(\tau) + \mu^{1/2}v_{1/2}(\tau) + \mu v_1(\tau) + \cdots$$

Let $\mu = 0$ in Equations (2.4). We have

$$u_0'' + \omega_1^2 u_0 = 0, \qquad v_0'' + \omega_2^2 v_0 = 0$$

Hence

$$u_0(\tau) = P_0\cos\omega_1\tau + Q_0\sin\omega_1\tau, \qquad v_0(\tau) = 0 \tag{3.2}$$

Equating to each other terms with $\mu^{\frac{1}{2}}$ in Equations (2.4), we have

$$u_{1/2}'' + \omega_1^2 u_{1/2} = -2\alpha_{1/2}u_0', \qquad v_{1/2}'' + \omega_2^2 v_{1/2} = 0$$

It follows from the periodicity conditions of the function $u_{1/2}(\tau)$ that

$$\alpha_{1/2} = 0 \tag{3.3}$$

where this is a double root. Thus we obtain

$$u_{1/2}(\tau) = P_{1/2}\cos\omega_1\tau + Q_{1/2}\sin\omega_1\tau, \qquad v_{1/2}(\tau) = 0 \tag{3.4}$$

Furthermore, in (2.4) we equate the terms with μ in first power. After certain transformations we get

$$u_1'' + \omega_1^2 u_1 = P_0 K_1(\tau) + Q_0 L_1(\tau)$$

$$v_1'' + \omega_2^2 v_1 = P_0\left(\frac{\partial\Phi_2}{\partial A_0}\right)_0 - Q_0\frac{1}{A_0\omega_1}\left(\frac{\partial\Phi_2}{\partial\tau}\right)_0 \tag{3.5}$$

Here

$$K_1(\tau) = \left(\frac{\partial\Phi_1}{\partial A_0}\right)_0 + 2\alpha_1\omega_1\sin\omega_1\tau, \qquad L_1(\tau) = -\frac{1}{A_0\omega_1}\left(\frac{\partial\Phi_1}{\partial\tau}\right)_0 - 2\alpha_1\omega_1\cos\omega_1\tau \tag{3.6}$$

The periodicity conditions for the function $u_1(\tau)$

$$\int_0^{T_1}[P_0K_1(\tau) + Q_0L_1(\tau)]\begin{pmatrix}\cos\omega_1\tau\\ \sin\omega_1\tau\end{pmatrix}d\tau = 0$$

lead to Equations

$$P_0\left(\omega_1\frac{dC_{11}^*}{dA_0} - 2\pi\alpha_1\right) = 0, \qquad Q_0(-2\pi\alpha_1) = 0$$

Two solutions are possible for these equations

$$\alpha_1 = \frac{1}{T_1}\frac{dC_{11}}{dA_0}, \qquad Q_0 = 0 \tag{3.7}$$

or

$$\alpha_1 = 0, \qquad P_0 = 0 \tag{3.8}$$

Here and in what follows the absence of the argument in the C_{12} functions or their derivatives means that these functions or their derivatives are evaluated at $\tau = T_1$.

The functions $u_1(\tau)$ is defined by Formula

$$u_1(\tau) = P_1 \cos \omega_1 \tau + Q_1 \sin \omega_1 \tau + P_0 \left[\frac{dC_{11}^*(\tau)}{dA_0} - \alpha_1 \tau \cos \omega_1 \tau \right] - $$
$$- Q_0 \left[\frac{1}{A_0 \omega_1} C_{11}^{*\prime}(\tau) + \alpha_1 \tau \sin \omega_1 \tau \right] \qquad (3.9)$$

Here the terms proportional to $\sin \omega_1 \tau$ with coefficients independent of τ, which must be placed in parentheses at P_0 and Q_0, are included in $Q_1 \sin \omega_1 \tau$.

The general solution of the second equation in (3.5) is

$$v_{1*}(\tau) = R_1 \cos \omega_2 \tau + S_1 \sin \omega_2 \tau + P_0 \frac{dC_{21}^*(\tau)}{dA_0} +$$
$$+ Q_0 \frac{1}{A_0 \omega_1} \left[\frac{1}{\omega_2} H_{21}^*(0) \sin \omega_2 \tau - C_{21}^{*\prime}(\tau) \right]$$

Choosing the arbitrary constants R_1 and S_1 in a definite manner, we can obtain a periodic solution with the period T_1

$$v_1(\tau) = P_0 \frac{dD_1^*(\tau)}{dA_0} - Q_0 \frac{1}{A_0 \omega_1} D_1^{*\prime}(\tau) \qquad (3.10)$$

Next, the equation for $u_{3/2}(\tau)$ is constructed. After a number of transformations we get

$$u_{3/2}'' + \omega_1^2 u_{3/2} = P_0 K_{3/2}(\tau) + Q_0 L_{3/2}(\tau) + P_{1/2} K_1(\tau) + Q_{1/2} L_1(\tau)$$

The following notation is used here:

$$K_{3/2}(\tau) = A_{1/2} \left(\frac{\partial^2 \Phi_1}{\partial A_0^2} \right)_0 + 2\alpha_{3/2} \omega_1 \sin \omega_1 \tau \qquad (3.11)$$

$$L_{3/2}(\tau) = - \frac{A_{1/2}}{A_0 \omega_1} \left[\left(\frac{\partial^2 \Phi_1}{\partial \tau \partial A_0} \right)_0 - \frac{1}{A_0} \left(\frac{\partial \Phi_1}{\partial \tau} \right)_0 \right] - 2\alpha_{3/2} \omega_1 \cos \omega_1 \tau$$

The periodicity conditions lead to Equations

$$P_0 \left(A_{1/2} \omega_1 \frac{d^2 C_{11}^*}{dA_0^2} - 2\pi \alpha_{3/2} \right) + P_{1/2} \left(\omega_1 \frac{dC_{11}^*}{dA_0} - 2\pi \alpha_1 \right) = 0$$
$$Q_0 \left(A_{1/2} \frac{\omega_1}{A_0} \frac{dC_{11}^*}{dA_0} - 2\pi \alpha_{3/2} \right) - Q_{1/2} 2\pi \alpha_1 = 0$$

If $\alpha_1 = \dfrac{1}{T_1} \dfrac{dC_{11}}{dA_0}$, then $Q_0 = 0$ and consequently

$$\alpha_{3/2} = \frac{1}{T_1} A_{1/2} \frac{d^2 C_{11}}{dA_0^2}, \qquad Q_{1/2} = 0 \qquad (3.12)$$

If $\alpha_1 = 0$, then $P_0 = 0$ and consequently

$$\alpha_{3/2} = \frac{1}{T_1} \frac{A_{1/2}}{A_0} \frac{dC_{11}}{dA_0} = 0, \qquad P_{1/2} = 0 \qquad (3.13)$$

since for $dC_{11} / dA_0 \neq 0$ we have $A_{1/2} = 0$.

Finally, the equation for $u_2(\tau)$ is constructed. Here it is assumed that $dC_{11} / dA_0 = 0$ and, consequently, $\alpha_1 = 0$. After quite unwieldy transformations there results

$$u_2'' + \omega_1^2 u_2 = \sum_{s=0,\ 1/2,\ 1} [P_s K_{2-s}(\tau) + Q_s L_{2-s}(\tau)]$$

Here

$$K_2(\tau) = \frac{1}{2} A_{1/2}^2 \left(\frac{\partial^3 \Phi_1}{\partial A_0^3} \right)_0 + A_1 \left(\frac{\partial^2 \Phi_1}{\partial A_0^2} \right)_0 + \frac{\partial H_{12}^*(\tau)}{\partial A_0} + 2\alpha_2 \omega_1 \sin \omega_1 \tau$$

$$L_2(\tau) = - \frac{1}{A_0 \omega_1} \left\{ \frac{1}{2} A_{1/2}^2 \left[\left(\frac{\partial^3 \Phi_1}{\partial \tau \partial A_0^2} \right)_0 - \frac{2}{A_0} \left(\frac{\partial^2 \Phi_1}{\partial \tau \partial A_0} \right)_0 + \frac{2}{A_0^2} \left(\frac{\partial \Phi_1}{\partial \tau} \right)_0 \right] + \right.$$
$$\left. + A_1 \left[\left(\frac{\partial^2 \Phi_1}{\partial \tau \partial A_0} \right)_0 - \frac{1}{A_0} \left(\frac{\partial \Phi_1}{\partial \tau} \right)_0 \right] + \frac{\partial H_{12}^*(\tau)}{\partial \tau} \right\} - 2\alpha_2 \omega_1 \cos \omega_1 \tau$$

The periodicity conditions lead to Equations

$$P_0\left(\frac{1}{2}A_{1/2}{}^2\omega_1\frac{d^3C_{11}{}^*}{dA_0{}^3}+A_1\omega_1\frac{d^2C_{11}{}^*}{dA_0{}^2}+\omega_1\frac{dC_{12}{}^*}{dA_0}-2\pi\alpha_2\right)+ \\ +P_{1/2}\left(A_{1/2}\omega_1\frac{d^2C_{11}{}^*}{dA_0{}^2}-2\pi\alpha_{3/2}\right)=0 \tag{3.14}$$

$$Q_0\left(\frac{1}{2}A_{1/2}{}^2\frac{\omega_1}{A_0}\frac{d^2C_{11}{}^*}{dA_0{}^2}+\frac{\omega_1}{A_0}C_{12}{}^*(T_1)-2\pi\alpha_2\right)-Q_{1/2}2\pi\alpha_{3/2}=0$$

For the first branch of the characteristic exponent we obtain

$$\alpha_2=\frac{1}{T_1}\left(\frac{1}{2}A_{1/2}{}^2\frac{d^3C_{11}}{dA_0{}^3}+A_1\frac{d^2C_{11}}{dA_0{}^2}+\frac{dM_2}{dA_0}\right) \tag{3.15}$$

For the second branch we have

$$\alpha_2=\frac{1}{T_1A_0}\left(\frac{1}{2}A_{1/2}{}^2\frac{d^2C_{11}}{dA_0{}^2}+M_2\right)=0 \tag{3.16}$$

since the expression in the parentheses is equal to zero in view of the equation [2] which determines the coefficient $A_{1/2}$.

The subsequent coefficients of the $\alpha^{(1)}$ expansion were not evaluated. Thus, we have

$$\alpha^{(1)}=\frac{1}{T_1}\left[\frac{dC_{11}}{dA_0}\mu+A_{1/2}\frac{d^2C_{11}}{dA_0{}^2}\mu^{3/2}+\left(\frac{1}{2}A_{1/2}{}^2\frac{d^3C_{11}}{dA_0{}^3}+A_1\frac{d^2C_{11}}{dA_0{}^2}+\frac{dM_2}{dA_0}\right)\mu^2+\cdots\right] \tag{3.17}$$

for the first branch and $\alpha^{(1)}=0$ for the second branch of the characteristic exponent corresponding to the critical roots of the fundamental equation.

4. For triple roots of Equation (2.1) it is possible to have periodic solutions of the system (1.8) with a period T_1 which are represented by series in integral powers of $\mu^{1/3}$. In this case the characteristic exponent $\alpha^{(1)}$, and the functions $u^{(1)}(\tau)$ and $v^{(1)}(\tau)$ can also be expanded in series in powers of $\mu^{1/3}$. For example,

$$\alpha^{(1)}=\alpha_{1/3}\mu^{1/3}+\alpha_{2/3}\mu^{2/3}+\alpha_1\mu+\cdots \tag{4.1}$$

Computing the coefficients of this series by the method analogous to the previous one, and taking into account the fact that in the present case

$$\frac{dC_{11}}{dA_0}=\frac{d^2C_{11}}{dA_0{}^2}=0$$

we will obtain for both branches of the characteristic exponent

$$\alpha_{1/3}=\alpha_{2/3}=\alpha_{4/3}=0 \tag{4.2}$$

For the first branch we will also have

$$\alpha_1=\frac{1}{T_1}\frac{dC_{11}}{dA_0},\qquad \alpha_{5/3}=\frac{1}{2T_1}A_{1/3}{}^2\frac{d^3C_{11}}{dA_0{}^3} \\ \alpha_2=\frac{1}{T_1}\left(\frac{1}{6}A_{1/3}{}^3\frac{d^4C_{11}}{dA_0{}^4}+A_{1/3}A_{2/3}\frac{d^3C_{11}}{dA_0{}^3}+\frac{dM_2}{dA_0}\right) \tag{4.3}$$

The subsequent coefficients of $\alpha_{n/3}$ were not computed.

5. Let us turn now to the calculation of the characteristic exponent $\alpha^{(2)}$ for the noncritical roots of the fundamental equation in system (1.8). We will investigate the case when the periodic solution of this system is expanded into a power series in $\mu^{1/2}$. Then

$$\alpha^{(2)}=\alpha_0+\alpha_{1/2}\mu^{1/2}+\alpha_1\mu+\ldots,\qquad \alpha_0=\pm i\omega_2 \tag{5.1}$$

The functions $u^{(2)}(\tau)$ and $v^{(2)}(\tau)$ are also expanded in series of $\mu^{1/2}$. These functions must be periodic with period T_1

From Equation (2.4) for $\mu = 0$ we obtain

$$u_0'' + 2\alpha_0 u_0' + (\omega_1^2 - \omega_2^2)u_0 = 0, \qquad v_0'' + 2\alpha_0 v_0' = 0 \tag{5.2}$$

We seek the solution of these equations in the form

$$u_0(\tau) = U_0 e^{im\tau}, \qquad v_0(\tau) = V_0 e^{in\tau}$$

Substituting these expressions into (5.2), we find

$$m_1 = \pm(\omega_1 - \omega_2), \quad m_2 = \mp(\omega_1 + \omega_2), \quad n_1 = 0, \quad n_2 = \mp 2\omega_2 \tag{5.3}$$

Thus, Equations (5.2) have the following solution satisfying the requirements of periodicity:

$$u_0(\tau) = 0, \qquad v_0(\tau) = V_0 \tag{5.4}$$

The consideration of the system of equations for $u_{1/2}(\tau)$ and $v_{1/2}(\tau)$ leads to the following results:

$$\alpha_{1/2} = 0, \qquad u_{1/2}(\tau) = 0, \qquad v_{1/2}(\tau) = V_{1/2} \tag{5.5}$$

Furthermore, we construct the equation for $v_1(\tau)$

$$v_1'' + 2\alpha_0 v_1' = \left[-2\alpha_0\alpha_1 + \left(\frac{\partial\Phi_2}{\partial z_2}\right)_0 + \alpha_0\left(\frac{\partial\Phi_2}{\partial z_2'}\right)_0\right] v_0 \tag{5.6}$$

The periodicity condition for the solution $v_1(\tau)$ is reduced to the absence of a constant term in the right-hand part of this equation. Hence we get

$$\alpha_1 = \frac{1}{2T_1}\int_0^{T_1}\left[\left(\frac{\partial\Phi_2}{\partial z_2'}\right)_0 \mp \frac{i}{\omega_2}\left(\frac{\partial\Phi_2}{\partial z_2}\right)_0\right] d\tau \tag{5.7}$$

Obviously, the same expression for α_1 will be obtained if the periodic solution expanded in powers of $\mu^{1/3}$ is considered.

6. The investigation of stability for the cases of double and triple roots of the amplitude equation for a quasi-linear self-contained system with a single degree of freedom was carried out in [4]. This work obtained the expansions for one root of the characteristic equation in powers of $\mu^{1/2}$ and $\mu^{1/3}$. The second root for the self-contained system is equal to unity. The characteristic exponents α used in the present paper and the roots of the characteristic equation ρ are related by

$$\alpha = T^{-1}\ln\rho \quad (T \text{ is the solution period}) \tag{6.1}$$

Present results show that for the system with two degrees of freedom there exist four branches of the characteristic exponent from which, in the case of incommensurable frequencies, one branch is real and nonzero in general, the second branch zero and two branches complex. Since the function $x_1(t)$ entering into the solution of system (1.1) is of the same form as the solution of a system with a single degree of freedom obtainable by neglecting the second equation in (1.1), the function $y_1(t)$ from the solution of the equations in variations for system (1.1) is of the same form as the solution of the equation in variations for a system with a single degree of freedom. Therefore, the first branches of the characteristic exponent for both systems possess identical forms of expansions in integral of fractional power of the small parameter. Second branches in both cases are zero.

The signs of the real parts of the characteristic exponents are determined for sufficiently small μ by the first, nonzero, coefficients of the exponent expansions. Then, one of the stability conditions coincides in form with the corresonding condition of stability for a system with a single degree of freedom if this condition $C_s(T_0)$ is everywhere replaced by $C_{1s}(T_1)$. The second condition of stability is the inequality

$$\int_0^{T_1}\left(\frac{\partial\Phi_2}{\partial z_2'}\right)_0 d\tau = \int_0^{T_1}\left(\frac{\partial F_2}{\partial x_2'}\right)_0 dt < 0 \tag{6.2}$$

This result can be easily generalized for the case of the self-contained quasi-linear system with n degrees of freedom when one frequency of the generating system is incommensurable with any other frequency. Also, in addition to the basic stability condition analogous to the condition for a single degree of freedom, there will esist $n - 1$ auxiliary conditions of the type (6.2).

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Translated by V.A.C.